Line-graph inertia of roses and generalized theta graphs

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Abstract

For a graph G, the adjacency inertia of the line graph $L(G)$ is determined by the number of eigenvalues of the signless Laplacian $Q(G)$ above, at, and below 2. We compute $\mathrm{In}(Q(G) - 2I)$, and hence $\mathrm{In}\,A(L(G))$, exactly, including every singular case, for rose graphs and generalized theta graphs.

Both computations follow from one reduction. Deleting the common vertices leaves disjoint paths; range–kernel elimination leaves their singular kernel directions and a residual scalar for a rose or a 2×2 matrix for a generalized theta. Terminal exchange fixes the latter's eigenvectors. The resulting formulas depend only on the path lengths modulo 4.

We obtain closed expressions for the full inertia, signature, and multiplicity $m_Q(G, 2)$. One theta mode is the rose scalar shifted by the second terminal's contribution; the other has no rose analogue. Further consequences include $m_Q(G, 2) \le c(G)$ for generalized thetas with at least three paths and an exact comparison with the conjectured bound $2s(L(G)) \le c(G) + 1$. Its slack grows linearly with the cyclomatic number on both classes, with equality only for cycles of length $1 \pmod 4$; the general conjecture is not proved. We also give a partial extension to bridgeless cacti. Exact finite computations check every formula branch, and the accompanying archive supplies source, scripts, frozen outputs, and build information.

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1 Introduction

For a real symmetric matrix M write $\mathrm{In}\,M = (n_+(M), n_0(M), n_-(M))$ and $\mathrm{sig}\,M = n_+(M) - n_-(M)$; the signature of a graph is the signature of its adjacency matrix. Line-graph signature converts into a threshold problem for the signless Laplacian $Q(G) = D(G) + A(G)$: the nonoriented incidence matrix B gives

$$Q(G) = BB^T, \qquad A(L(G)) + 2I = B^T B, \tag{1}$$

so the inertia of $A(L(G))$ is determined by the distribution of the spectrum of $Q(G)$ about 2. This identity, with path-response and congruence techniques, underlies general line-graph signature bounds and recent counterexample and transfer constructions [1, 2, 3].

Exact answers of this kind are known for a few classes. Ma, Wong and Zhu determine the positive and negative inertia index of line graphs of trees [4]. Li, Fan and Su determine the nullity of the line graph of a unicyclic graph of depth one [5]. Wang and Fan bound the signature of a line graph by counts of cycles of length $4k+3$ and $4k+5$ [1]. Beyond trees the picture is fragmentary.

What this paper does

We compute $\mathrm{In}(Q(G)-2I)$ completely for two classes of graphs in which several cycles share structure.

- A *rose* $R(\ell_1,\ldots,\ell_t)$ is t cycles identified at one common vertex. Deleting that vertex leaves t paths, each meeting the deleted vertex at both of its ends.
- A *generalized theta graph* $\Theta(\ell_1,\ldots,\ell_t)$ consists of t internally disjoint paths joining one pair of vertices. Deleting that pair leaves t paths, each meeting one deleted vertex at one end and the other at the other end.

Figure 1 shows both classes and, in the lower row, the effect of deleting the shared vertices. The difference the paper turns on is visible there: in a rose the two ends of each remaining path return to the same deleted vertex, whereas in a generalized theta they go to different ones.

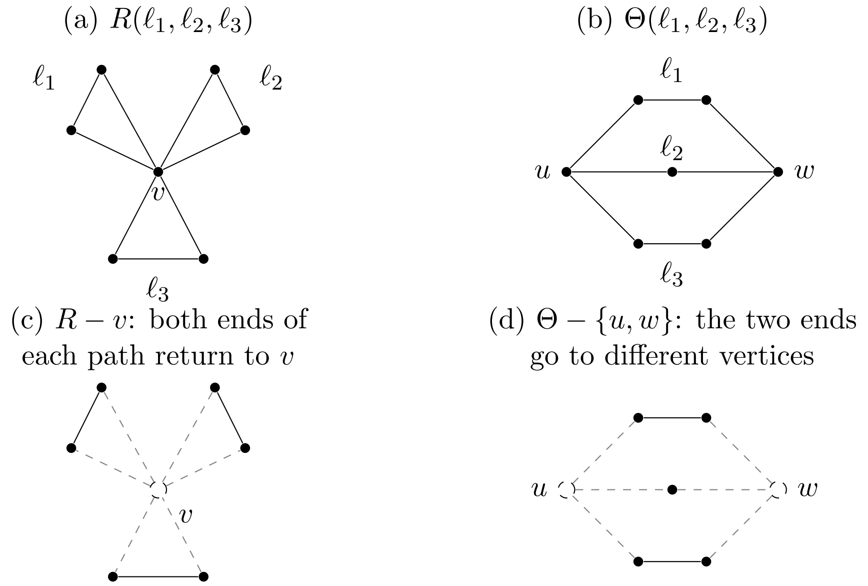

Figure 1: The two classes, and the effect of deleting the shared vertices. The drawings are schematic, and ℓ_i denotes the number of edges of the i-th cycle or path. Filled vertices remain, dashed circles are deleted, and dashed edges are the ones lost. In both lower panels the solid subgraph is a disjoint union of paths, which is the hypothesis of Proposition 4.3. What differs is where the ends of those paths attach: to one deleted vertex in (c), to two in (d). The middle path in (d) has length 2, so it consists of a single vertex adjacent to both terminals.

The two computations are neither unrelated nor identical. Both are instances of one reduction, applied to a set B of deleted vertices with $|B|=1$ for a rose and $|B|=2$ for a theta. The size of B is what drives the difference: it fixes how many ends of each path can attach to distinct vertices, and hence how many eigenvectors the reduced matrix has to offer. Those consequences are spelled out below.

The reduction produces, in both cases,

$$\operatorname{In}(Q(G)-2I) = (q,0,q) + \operatorname{In}\big(Z^T P' Z\big) + \big(0,\ \kappa,\ 0\big), \tag{2}$$

Here $q = \sum_i \lfloor (\ell_i - 1)/2 \rfloor$ accounts for the invertible part of the path blocks, and P' is a reduced symmetric matrix of order $|B|$. The columns of Z span a subspace of $\mathbb{R}^{|B|}$ determined by the even-length blocks, and κ counts the kernel directions of the blocks that are not adjacent to B.

Each even length contributes exactly one kernel direction. If that direction is adjacent to B, it removes one dimension from Z and contributes one positive and one negative eigenvalue.

For a rose, P' is the scalar

$$P' = 2x, \qquad x = n_0 + 2n_1 - 1. \tag{3}$$

For a generalized theta, reversing all paths and exchanging u with w is an automorphism, so P' has equal diagonal entries and $z_\pm = (1, \pm 1)^T$ are eigenvectors; the corresponding eigenvalues are

$$x_+ = n_0 + 2n_1 - 2, \qquad x_- = n_2 + 2n_3 - 2. \tag{4}$$

The relation between (3) and (4) is the point of the comparison. Contracting the theta coupling against z_+ gives $C_i z_+ = e_1 + e_{m_i}$. That is exactly the vector against which a rose petal is contracted. The two sums over blocks are therefore the same function of the residue counts, so x and x_+ can differ only through the contribution of the deleted vertices. They differ by 1: a single centre contributes $2t-2$, whereas a pair contributes $2t-4$ in the z_+ direction. The eigenvalue x_- has no rose analogue, because $\mathbb{R}^1$ contains no vector orthogonal to $z = 1$. This also explains a feature of the rose case that is opaque in isolation: that lengths $\equiv 0 \pmod 4$ leave the centre uncoupled while lengths $\equiv 2 \pmod 4$ do not. A rose sees only the z_+ direction, and the kernel directions of the 0 mod 4 blocks are orthogonal to it; in a theta the same directions act visibly on z_-. Proposition 7.1 makes this precise.

Results

Section 2 fixes notation and records the incidence transfer lemma. Section 3 computes, for a path block and both signs $s = \pm 1$, the quantity $b^T A^+ b$ with $b = e_1 + s\, e_m$, including the singular cases. Section 4 gives the two lemmas of linear algebra that make the elimination valid when the block matrix is singular, and assembles them into Proposition 4.3. Section 5 applies it with $|B| = 1$ and obtains the rose theorem; Section 6 applies it with $|B| = 2$ and proves the theta theorem. Section 7 compares the two reduced matrices and specializes the theta theorem to the rose theorem along the cycles. Section 8 treats the multiplicity $m_Q(G, 2)$; Section 9 the position of both classes relative to the conjectured bound $2s(L(G)) \le c(G)+1$, which remains open and is not a claim of this paper. Section 10 proves a partial extension to bridgeless cacti and identifies the obstruction to extending it, and Section 11 records the exact verification.

2 Preliminaries

All graphs are finite and simple. For a connected graph G the cyclomatic number is $c(G) = |E(G)| - |V(G)| + 1$.

Definition 2.1. For $t \geq 1$ and integers $\ell_i \geq 3$, the *rose* $R(\ell_1, \ldots, \ell_t)$ is formed from cycles $C_{\ell_1}, \ldots, C_{\ell_t}$ by identifying one vertex of every cycle and making no other identification. The identified vertex is the *centre* and the cycles are the *petals*.

For $t \geq 2$ and integers $\ell_i \geq 1$, the *generalized theta graph* $\Theta(\ell_1, \ldots, \ell_t)$ consists of two distinct vertices u, w, the *terminals*, joined by t internally disjoint paths of edge lengths $\ell_1, \ldots, \ell_t$. It is simple if and only if at most one ℓ_i equals 1, which we always assume.

Our convention explicitly includes $t = 2$ among generalized theta graphs. Thus $c(R) = t$ and $c(\Theta) = t - 1$. A rose with $t = 1$ and a theta with $t = 2$ are both cycles, and this is the only overlap of the two classes. Throughout, for either class, n_r is the number of indices i with $\ell_i \equiv r \pmod 4$, so $n_0 + n_1 + n_2 + n_3 = t$, and

$$q = \sum_{i=1}^{t} \left\lfloor \frac{\ell_i - 1}{2} \right\rfloor, \qquad \delta = \mathbf{1}_{\{\ell_i = 1 \text{ for some } i\}} \in \{0, 1\}. \tag{5}$$

Only a theta can have $\delta = 1$. We write $(y)_+ = \max(y, 0)$, and $\mathbf{1}_{\{S\}}$ for the indicator of a statement S, equal to 1 when S holds and 0 otherwise.

Lemma 2.2 (Incidence transfer)**.** *Let G be a connected simple graph with n vertices, m edges and cyclomatic number $c = m - n + 1$, and put $M = Q(G) - 2I$. Then*

$$\operatorname{In} A(L(G)) = \big(n_+(M),\, n_0(M),\, n_-(M) + c - 1\big), \qquad s(L(G)) = \operatorname{sig} M - c + 1.$$

Proof. By (1) the nonzero spectra of BB^T and B^TB coincide with multiplicity. Let $\beta = \dim\ker Q(G)$, which is 1 if G is bipartite and 0 otherwise. The β zero eigenvalues of $Q(G)$ give eigenvalues -2 of M. Since $\operatorname{rank} B = n - \beta$, the matrix B^TB has $m - n + \beta$ zero eigenvalues, which become -2 in $A(L(G))$. The negative index therefore increases by $(m - n + \beta) - \beta = c - 1$, while the positive and zero indices are unchanged. ☐

Remark 2.3. Since $n_0(Q(G) - 2I) = m_Q(G, 2)$, every nullity statement below is simultaneously a statement about the multiplicity of the signless-Laplacian eigenvalue 2.

3 Path blocks

Both classes reduce to the same local computation. Deleting B leaves paths, and a path on m vertices is adjacent to B only at its two ends. For $s \in \{+1, -1\}$ put

$$b^s = e_1 + s\, e_m \in \mathbb{R}^m.$$

Only b^+ occurs for a rose, both ends of a petal being adjacent to the same centre; both b^+ and b^- occur for a theta, whose two ends are adjacent to different vertices of B.

Lemma 3.1 (Path block)**.** *Let $\ell \geq 2$, $m = \ell - 1$ and $A = A(P_m)$. Then* $\operatorname{In} A(P_m) = \big(\lfloor m/2 \rfloor,\ \mathbf{1}_{\{m \text{ odd}\}},\ \lfloor m/2 \rfloor\big)$*, and:*

(i) *If ℓ is odd then A is invertible, $(A^{-1})_{11} = (A^{-1})_{mm} = 0$ and $(A^{-1})_{1m} = \sigma$ with $\sigma = -1$ for $\ell \equiv 1 \pmod 4$ and $\sigma = +1$ for $\ell \equiv 3 \pmod 4$. Hence $b^{sT} A^{-1} b^s = 2s\sigma$ for both signs.*

(ii) *If ℓ is even then $\dim\ker A = 1$, the kernel is spanned by a vector z with $z_1 = 1$ and $z_m = \varepsilon$, where $\varepsilon = -1$ for $\ell \equiv 0 \pmod 4$ and $\varepsilon = +1$ for $\ell \equiv 2 \pmod 4$, and*

$$b^s \in \operatorname{Range} A \iff s = -\varepsilon, \qquad \textit{in which case} \qquad b^{sT} A^+ b^s = 0.$$

For $s = \varepsilon$ one has $b^s \notin \operatorname{Range} A$.

Proof. The eigenvalues of $A(P_m)$ are $2\cos(\pi j/(m+1))$, $1 \le j \le m$, which gives the inertia.

Let m be even. Then $\det A(P_m) = (-1)^{m/2}$, from $\det A(P_k) = -\det A(P_{k-2})$. Cofactor expansion gives $(A^{-1})_{11} = \det A(P_{m-1})/\det A(P_m) = 0$ since $m-1$ is odd, and symmetrically $(A^{-1})_{mm} = 0$. Deleting row m and column 1 from the tridiagonal matrix leaves a matrix with entries $A_{i,j+1}$, which is upper triangular with unit diagonal and determinant 1; hence $(A^{-1})_{1m} = (-1)^{1+m}/(-1)^{m/2} = (-1)^{m/2+1}$, equal to -1 when $\ell = m+1 \equiv 1 \pmod 4$ and to $+1$ when $\ell \equiv 3 \pmod 4$. Then $b^{sT}A^{-1}b^s = (A^{-1})_{11} + 2s(A^{-1})_{1m} + s^2(A^{-1})_{mm} = 2s\sigma$.

Let m be odd. Then $\ker A = \langle z\rangle$ with $z = (1, 0, -1, 0, 1, \dots, (-1)^{(m-1)/2})^T$, so $z_1 = 1$ and $z_m = (-1)^{(m-1)/2} = \varepsilon$, which is -1 for $\ell \equiv 0 \pmod 4$ and $+1$ for $\ell \equiv 2 \pmod 4$. As A is symmetric, $\operatorname{Range} A = (\ker A)^\perp$, and $z^T b^s = 1 + s\varepsilon$ vanishes exactly for $s = -\varepsilon$, while for $s = \varepsilon$ it equals 2. For $s = -\varepsilon$ let $Ay = b^s$; the response $b^{sT}y$ is independent of the choice of y because $b^s \perp \ker A$, so $b^{sT}A^+b^s = b^{sT}y$. The interior equations of $Ay = b^s$ give $y_{j-1} + y_{j+1} = 0$ for $2 \le j \le m-1$; stepping by two from index 1 to the odd index m yields $y_m = \varepsilon y_1 = -s\, y_1$, whence $b^{sT}y = y_1 + s\, y_m = 0$. ☐

Remark 3.2 (The two singular branches are one phenomenon seen twice)*.* Part (ii) is symmetric under $(\ell, s) \mapsto (\ell+2, -s)$: for a block of even length exactly one of the two signs gives $b^s \in \operatorname{Range} A$, and which one is decided by $\ell \bmod 4$. In a rose only $s = +1$ occurs, so this appears as an asymmetry between 0 and 2 mod 4; in a theta both signs occur and the symmetry is visible.

Remark 3.3 (The direct edge needs no case of its own)*.* A theta path of length 1 has no internal block; it contributes $\delta = 1$ to P_0 and nothing to the sum over blocks. By part (i) a path with $\ell \equiv 1 \pmod 4$ and $\ell \ge 5$ contributes $-b^{sT}A^{-1}b^s = +2s$ for the sign s, which is exactly what the δ entry contributes. So a length-1 path may be counted in n_1 with no correction term, for both signs.

4 Singular elimination and the reduction

Both classes give a matrix of the form

$$M = Q(G) - 2I = \begin{pmatrix} P_0 & C^T \\ C & A \end{pmatrix}, \qquad A = \bigoplus_i A(P_{m_i}), \tag{6}$$

with P_0 of order $p \in \{1, 2\}$ on the boundary and C the coupling. As soon as some ℓ_i is even, A is singular. An ordinary Schur complement is then unavailable, and only the range directions may be eliminated.

Lemma 4.1 (Range–kernel elimination)**.** *Let $M = \begin{pmatrix} P_0 & C^T \\ C & A \end{pmatrix}$ with A real symmetric of order N, and let C_R, C_K be the components of C under the orthogonal splitting $\mathbb{R}^N = \operatorname{Range} A \oplus \ker A$. Then M is congruent to*

$$\left(A|_{\operatorname{Range} A}\right) \;\oplus\; \begin{pmatrix} P' & C_K^T \\ C_K & 0 \end{pmatrix}, \qquad P' = P_0 - C^T A^+ C,$$

with A^+ the Moore–Penrose inverse.

Proof. In an orthonormal basis adapted to the splitting, $A = \widehat{A} \oplus 0$ with $\widehat{A} = A|_{\operatorname{Range} A}$ invertible and $C = \binom{C_R}{C_K}$. The congruence by $S = \begin{pmatrix} I & 0\ 0 \\ -\widehat{A}^{-1}C_R & I\ 0 \\ 0 & 0\ I \end{pmatrix}$ replaces P_0 by $P_0 -$

$C_R^T\widehat{A}^{-1}C_R$, annihilates the C_R coupling, and changes neither $\widehat{A}$ nor C_K; each of the six blocks is checked directly. Finally $C_R^T\widehat{A}^{-1}C_R = C^TA^+C$, because in this basis $A^+ = \widehat{A}^{-1}\oplus 0$. $\square$

Lemma 4.2 (Saddle-point inertia)**.** *Let P be real symmetric of order p and V real of size $k\times p$ with* $\operatorname{rank} V=\rho$. *For any Z whose columns are a basis of* $\ker V$*,*

$$\operatorname{In}\begin{pmatrix} P & V^T \\ V & 0 \end{pmatrix} = (\rho,\, k-\rho,\, \rho) + \operatorname{In}\bigl(Z^TPZ\bigr).$$

Proof. Let R be invertible with RV in reduced row echelon form. The congruence by $I\oplus R$ replaces V by RV, whose last $k-\rho$ rows vanish; those coordinates become zero rows and columns and contribute $(0,k-\rho,0)$. So assume $\operatorname{rank} V=k=\rho$. Split $\mathbb{R}^p=\ker V\oplus(\ker V)^\perp$; in adapted coordinates $V=(0\ \ W)$ with W invertible of order ρ, and the matrix becomes

$$\begin{pmatrix} Z^TPZ & X^T & 0 \\ X & Y & W^T \\ 0 & W & 0 \end{pmatrix}.$$

The trailing block $J=\left(\begin{smallmatrix} Y & W^T \\ W & 0 \end{smallmatrix}\right)$ is invertible with $J^{-1}=\left(\begin{smallmatrix} 0 & W^{-1} \\ W^{-T} & -W^{-T}YW^{-1} \end{smallmatrix}\right)$, so $(J^{-1})_{11}=0$ and $\operatorname{In} J=(\rho,0,\rho)$. The Schur complement of J is therefore $Z^TPZ - X^T(J^{-1})_{11}X = Z^TPZ$. $\square$

We can now state the reduction once for both classes. For a rose the set B below is the centre, with $p=1$; for a generalized theta it is the pair of terminals, with $p=2$. A theta path of length 1 is the degenerate block with $m=0$, handled by Remark 3.3.

Proposition 4.3 (Reduction)**.** *Let G be connected and let $B\subseteq V(G)$ with $|B|=p$ be such that every component of $G-B$ is a path whose two formal ends are adjacent to B and which has no other neighbour in B. For a one-vertex path the two formal ends coincide, but the two end incidences are retained. Let the components have $\ell_1-1,\dots,\ell_t-1$ vertices. For each even ℓ_i let $v_i\in\mathbb{R}^p$ be the trace on B of the kernel vector of the i-th block, normalised by $z_{i,1}=1$, and let V be the $(n_0+n_2)\times p$ matrix of those rows. Then, with $P'=P_0-C^TA^+C$, $\rho=\operatorname{rank} V$, and Z a basis of* $\ker V$*,*

$$\operatorname{In}(Q(G)-2I) = (q,0,q) + \bigl(\rho,\ n_0+n_2-\rho,\ \rho\bigr) + \operatorname{In}\bigl(Z^TP'Z\bigr).$$

Proof. Apply Lemma 4.1 to (6). By Lemma 3.1, $\operatorname{In}(A_i|_{\operatorname{Range} A_i}) = (\lfloor m_i/2\rfloor, 0, \lfloor m_i/2\rfloor)$ in both parities and $\lfloor m_i/2\rfloor = \lfloor(\ell_i-1)/2\rfloor$, while a block with $m_i=0$ contributes nothing and also contributes 0 to q; summing over blocks gives $(q,0,q)$. Again by Lemma 3.1, $\ker A=\bigoplus_i \ker A_i$ has exactly one dimension per even ℓ_i, so n_0+n_2 in all, and the corresponding rows of C_K are the traces v_i. Lemma 4.1 delivers those rows with respect to an orthonormal kernel basis, hence as $v_i/\|z_i\|$; positive row scaling is the congruence by $I\oplus\operatorname{diag}(\|z_i\|^{-1})$, and Lemma 4.2 depends on V only through $\operatorname{rank} V$ and $\ker V$, so the normalisation is immaterial. Now apply Lemma 4.2. $\square$

Everything that follows is the evaluation of V and of $Z^TP'Z$ at $p=1$ and $p=2$.

5 One boundary vertex: roses

Order the coordinates of $R = R(\ell_1, \dots, \ell_t)$ with the centre first. The centre has degree $2t$, so $P_0 = (2t-2)$, a scalar, and petal i becomes a path on $m_i = \ell_i - 1$ vertices whose two ends are both adjacent to the centre. The coupling column of petal i is therefore b^+, and $\mathbb{R}^1$ is spanned by $z = 1$.

Theorem 5.1 (Rose inertia)**.** *Let $R = R(\ell_1, \dots, \ell_t)$ and $x = n_0 + 2n_1 - 1$. Put*

$$\Phi = \begin{cases} (1,0,1), & n_2 > 0, \\ \big(\mathbf{1}_{\{x>0\}}, \mathbf{1}_{\{x=0\}}, \mathbf{1}_{\{x<0\}}\big), & n_2 = 0. \end{cases}$$

Then

$$\mathrm{In}(Q(R) - 2I) = (q,0,q) + \Phi + \big(0,\ n_0 + n_2 - \mathbf{1}_{\{n_2>0\}},\ 0\big),$$

and $\mathrm{In}\,A(L(R))$ is the same triple with the negative index increased by $t-1$. Explicitly, for $n_2 > 0$,

$$\mathrm{In}(Q(R) - 2I) = (q+1,\ n_0+n_2-1,\ q+1), \qquad \mathrm{In}\,A(L(R)) = (q+1,\ n_0+n_2-1,\ q+t),$$

and for $n_2 = 0$,

$$\mathrm{In}(Q(R) - 2I) = \big(q + \mathbf{1}_{\{x>0\}},\ n_0 + \mathbf{1}_{\{x=0\}},\ q + \mathbf{1}_{\{x<0\}}\big).$$

Proof. Apply Proposition 4.3 with $p = 1$. Here V is a column of the scalars $v_i = z_i^T b^+ = 1 + \varepsilon_i$, which vanishes for $\ell_i \equiv 0 \pmod 4$ and equals 2 for $\ell_i \equiv 2 \pmod 4$ by Lemma 3.1. Hence $\mathrm{rank}\,V = \mathbf{1}_{\{n_2>0\}}$, and $\ker V = \mathbb{R}^1$ when $n_2 = 0$ and 0 otherwise.

If $n_2 > 0$ then Z is empty and the last term of Proposition 4.3 vanishes, giving $(q+1,\ n_0+n_2-1,\ q+1)$.

If $n_2 = 0$ then $Z = 1$ and by Lemma 3.1 every petal is compatible with b^+, so

$$P' = (2t-2) - \sum_i b^{+T} A_i^{-1} b^+ = 2t - 2 + 2n_1 - 2n_3 = 2(n_0 + 2n_1 - 1) = 2x,$$

using $t = n_0 + n_1 + n_3$ and $\delta = 0$. Its sign supplies the three indicators. Finally $c(R) = t$, so Lemma 2.2 adds $t-1$ to the negative index. ☐

Corollary 5.2 (Rose signature)**.** *$s(L(R)) = 1 - t + \mathbf{1}_{\{n_2=0\}}\,\mathrm{sign}(x)$. In particular $s(L(R)) \le 0$ whenever $t \ge 2$.*

Theorem 5.1 and Corollary 5.2 give a complete inertia formula for the rose class, including every singular branch and every zero multiplicity. Published antecedents cover proper parts of the class, and Section 12 records which; the derivation from Proposition 4.3 is also what makes the comparison of Section 7 possible.

6 Two boundary vertices: generalized thetas

Order the coordinates of $\Theta = \Theta(\ell_1, \dots, \ell_t)$ with u, w first. Both terminals have degree t, so

$$P_0 = \begin{pmatrix} t-2 & \delta \\ \delta & t-2 \end{pmatrix},$$

and the coupling block of path i is the $m_i \times 2$ matrix $C_i = (e_1 \ \ e_{m_i})$: the first internal vertex is adjacent to u, the last to w. A path of length 2 has $m_i = 1$, so $A_i = (0)$ and $e_1 = e_{m_i}$: its unique internal vertex is adjacent to both terminals.

Reversing all paths and exchanging u with w is an automorphism of Θ whose permutation matrix splits as $\Pi = \Pi_B \oplus \Pi_{\text{int}}$ with $\Pi_B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. From $\Pi M \Pi^T = M$ we get $\Pi_{\text{int}} A \Pi_{\text{int}}^T = A$, hence $\Pi_{\text{int}} A^+ \Pi_{\text{int}}^T = A^+$, and $\Pi_B P_0 \Pi_B^T = P_0$; therefore $\Pi_B P' \Pi_B^T = P'$, so P' has equal diagonal entries. Consequently

$$z_+ = (1,1)^T, \qquad z_- = (1,-1)^T$$

are eigenvectors of P', for every choice of the lengths. Note that $C_i z_s = e_1 + s\, e_{m_i} = b^s$, the vector of Section 3.

Lemma 6.1 (Eigenvalues of P'). *With $x_+ = n_0 + 2n_1 - 2$ and $x_- = n_2 + 2n_3 - 2$,*

$$z_+^T P' z_+ = 2x_+ \quad \textit{whenever } n_2 = 0, \qquad z_-^T P' z_- = 2x_- \quad \textit{whenever } n_0 = 0.$$

Since $\|z_+\|^2 = \|z_-\|^2 = 2$, the corresponding eigenvalue of P' on a surviving direction is x_+ or x_-, respectively; the displayed quadratic form is twice that eigenvalue.

Proof. Since $C_i z_s = b^s$, we have $z_s^T C^T A^+ C z_s = \sum_i b^{sT} A_i^+ b^s$ whenever every summand is defined, that is whenever $b^s \in \operatorname{Range} A_i$ for every i. By Lemma 3.1 this is exactly $n_2 = 0$ for $s = +1$ and $n_0 = 0$ for $s = -1$.

For $s = +1$ and $n_2 = 0$ the responses are -2 for $\ell_i \equiv 1$ with $\ell_i \geq 5$, $+2$ for $\ell_i \equiv 3$, and 0 for $\ell_i \equiv 0$; there are $n_1 - \delta$ of the first kind by Remark 3.3. With $z_+^T P_0 z_+ = 2(t-2) + 2\delta$ and $t = n_0 + n_1 + n_3$,

$$z_+^T P' z_+ = 2(t-2) + 2\delta + 2(n_1 - \delta) - 2n_3 = 2(n_0 + 2n_1 - 2) = 2x_+.$$

For $s = -1$ and $n_0 = 0$ the responses are $+2$, -2 and 0 for $\ell_i \equiv 1$ with $\ell_i \geq 5$, $\ell_i \equiv 3$ and $\ell_i \equiv 2$ respectively; with $z_-^T P_0 z_- = 2(t-2) - 2\delta$ and $t = n_1 + n_2 + n_3$ this gives $2(n_2 + 2n_3 - 2) = 2x_-$. $\square$

Theorem 6.2 (Theta inertia). *Let $\Theta = \Theta(\ell_1, \ldots, \ell_t)$ be simple, $t \geq 2$. Put*

$$\Phi_+ = \begin{cases} (1,0,1), & n_2 > 0, \\ \big(\mathbf{1}_{\{x_+>0\}}, \mathbf{1}_{\{x_+=0\}}, \mathbf{1}_{\{x_+<0\}}\big), & n_2 = 0, \end{cases}$$

$$\Phi_- = \begin{cases} (1,0,1), & n_0 > 0, \\ \big(\mathbf{1}_{\{x_->0\}}, \mathbf{1}_{\{x_-=0\}}, \mathbf{1}_{\{x_-<0\}}\big), & n_0 = 0. \end{cases}$$

Then

$$\operatorname{In}(Q(\Theta) - 2I) = (q, 0, q) + \Phi_+ + \Phi_- + \big(0,\ n_0 + n_2 - \mathbf{1}_{\{n_0>0\}} - \mathbf{1}_{\{n_2>0\}},\ 0\big),$$

and $\operatorname{In} A(L(\Theta))$ is the same triple with the negative index increased by $t-2$. Explicitly, $\operatorname{In}(Q(\Theta) - 2I)$ equals

$$\begin{array}{ll} (q+2,\ n_0 + n_2 - 2,\ q+2), & n_0 > 0,\ n_2 > 0, \\ \big(q + 1 + \mathbf{1}_{\{x_+>0\}},\ n_0 - 1 + \mathbf{1}_{\{x_+=0\}},\ q + 1 + \mathbf{1}_{\{x_+<0\}}\big), & n_0 > 0,\ n_2 = 0, \\ \big(q + 1 + \mathbf{1}_{\{x_->0\}},\ n_2 - 1 + \mathbf{1}_{\{x_-=0\}},\ q + 1 + \mathbf{1}_{\{x_-<0\}}\big), & n_0 = 0,\ n_2 > 0, \\ \big(q + \mathbf{1}_{\{x_+>0\}} + \mathbf{1}_{\{x_->0\}},\ \mathbf{1}_{\{x_+=0\}} + \mathbf{1}_{\{x_-=0\}},\ q + \mathbf{1}_{\{x_+<0\}} + \mathbf{1}_{\{x_-<0\}}\big), & n_0 = n_2 = 0. \end{array}$$

Proof. Apply Proposition 4.3 with $p = 2$. The rows of V are $v_i = (1, \varepsilon_i)$ with $\varepsilon_i = -1$ for $\ell_i \equiv 0 \pmod 4$ and $+1$ for $\ell_i \equiv 2 \pmod 4$. Since $v_i \cdot z_+ = 1 + \varepsilon_i$ and $v_i \cdot z_- = 1 - \varepsilon_i$,

$$\operatorname{rank} V = \mathbf{1}_{\{n_0>0\}} + \mathbf{1}_{\{n_2>0\}}, \qquad \ker V = \begin{cases} \mathbb{R}^2, & n_0 = n_2 = 0, \\ \langle z_+\rangle, & n_0 > 0,\ n_2 = 0, \\ \langle z_-\rangle, & n_0 = 0,\ n_2 > 0, \\ 0, & n_0 > 0,\ n_2 > 0. \end{cases}$$

So a kernel direction coming from a length $\equiv 0 \pmod 4$ removes z_- from $\ker V$, and one coming from a length $\equiv 2 \pmod 4$ removes z_+.

If $n_0, n_2 > 0$ then Z is empty, giving the first case and, in particular, an answer independent of n_1 and n_3. If exactly one of n_0, n_2 is positive then Z is the surviving eigenvector and Lemma 6.1 supplies its value, whose sign gives the indicators. If $n_0 = n_2 = 0$ then $\operatorname{rank} V = 0$ and $Z = I_2$; every A_i is invertible, so by Lemma 3.1 and Remark 3.3 the matrix P' has diagonal $t - 2$ and off-diagonal $\delta - \sum_{\ell_i \geq 3} \sigma_i = n_1 - n_3$, whence eigenvalues $(t-2) \pm (n_1 - n_3) = 2n_1 - 2 = x_+$ and $2n_3 - 2 = x_-$ using $t = n_1 + n_3$, in agreement with Lemma 6.1. Finally $c(\Theta) = t - 1$, so Lemma 2.2 adds $t - 2$. □

Corollary 6.3 (Theta signature). *$s(L(\Theta)) = 2 - t + \mathbf{1}_{\{n_2=0\}} \operatorname{sign}(x_+) + \mathbf{1}_{\{n_0=0\}} \operatorname{sign}(x_-)$. In particular $s(L(\Theta)) \leq 1$, and $s(L(\Theta)) \leq 0$ for $t \geq 3$.*

Proof. The pairs contributed by $\operatorname{rank} V$, the term $(q, 0, q)$ and the uncoupled kernel directions all have signature 0, and an indicator triple has signature $\operatorname{sign}(x_\pm)$; subtract $c(\Theta) - 1 = t - 2$. The bracketed sum is at most 2, and equals 2 only if $n_0 = n_2 = 0$ with $n_1 \geq 2$ and $n_3 \geq 2$, forcing $t \geq 4$. □

7 Comparison of the two reduced matrices

Proposition 7.1. *Let $\ell_1, \ldots, \ell_t$ satisfy $\ell_i \geq 3$ and $n_2 = 0$, so that both $R(\ell_1, \ldots, \ell_t)$ and $\Theta(\ell_1, \ldots, \ell_t)$ are defined and $b^+ \in \operatorname{Range} A_i$ for every block. Then the two sums over blocks coincide,*

$$\sum_i b^{+T} A_i^+ b^+ = -2n_1 + 2n_3$$

in both classes, and the two reduced quantities differ by exactly 2:

$$P'_{rose} = 2(n_0 + 2n_1 - 1) = 2x, \qquad z_+^T P'_\Theta z_+ = 2(n_0 + 2n_1 - 2) = 2x_+, \qquad P'_{rose} - z_+^T P'_\Theta z_+ = 2.$$

The discrepancy comes entirely from P_0: a rose centre has degree $2t$ and contributes $2t - 2$, whereas a theta contributes $z_+^T P_0 z_+ = 2(t-2)$. Moreover x_- has no rose analogue, since $\mathbb{R}^1$ contains no vector orthogonal to $z = 1$.

Proof. The rose coupling column of block i is b^+ and the theta coupling satisfies $C_i z_+ = b^+$, so both response sums are $\sum_i b^{+T} A_i^+ b^+$, evaluated by Lemma 3.1 as $-2n_1 + 2n_3$ when $n_2 = 0$. Subtracting from $2t - 2$ and from $2(t-2)$ respectively, and using $t = n_0 + n_1 + n_3$, gives the two displayed values, whose difference is $(2t-2) - (2t-4) = 2$. The last sentence is the statement that $\dim \mathbb{R}^1 = 1$ admits no vector orthogonal to $z = 1$. □

Proposition 7.1 accounts for the asymmetry noted in Remark 3.2. A rose contracts only against $z = 1$, so the kernel direction of a 0 mod 4 petal, being orthogonal to it, does not appear and the petal is uncoupled; that of a 2 mod 4 petal is not orthogonal and the petal is coupled. In a theta both kernel directions act, one on z_+ and the other on z_-.

The two classes meet along the cycles, where the same graph admits both descriptions, so the two theorems must agree there. The agreement is automatic; what it forces is an identity between two expressions that do not look equal.

Proposition 7.2 (Specialization along the cycles)**.** *Let $\ell_1, \ell_2 \geq 1$ with at most one equal to 1, and put $L = \ell_1 + \ell_2$. Then $\Theta(\ell_1, \ell_2) \cong C_L \cong R(L)$, so Theorem 6.2 at $t = 2$ and Theorem 5.1 at $t = 1$ evaluate the inertia of one and the same matrix and therefore agree. Consequently the two right-hand sides are equal as arithmetic expressions; in particular, writing q_Θ and q_R for the two values of q,*

$$q_R = \begin{cases} q_\Theta, & \ell_1, \ell_2 \text{ both odd}, \\ q_\Theta + 1, & \text{otherwise}, \end{cases}$$

and the discrepancy is absorbed by the remaining two terms.

Proof. The isomorphisms are immediate from Definition 2.1, and inertia is an invariant of the matrix, not of the route used to compute it; the two theorems are two evaluations of $\mathrm{In}(Q(C_L) - 2I)$. For the displayed identity, if ℓ_1, ℓ_2 are both odd then $\lfloor(\ell_i - 1)/2\rfloor = (\ell_i - 1)/2$ and L is even, so $q_\Theta = (L-2)/2 = \lfloor(L-1)/2\rfloor = q_R$. If both are even then $\lfloor(\ell_i - 1)/2\rfloor = (\ell_i - 2)/2$ and L is even, so $q_\Theta = (L-4)/2 = q_R - 1$. If exactly one is even then L is odd and $q_\Theta = (L-3)/2 = (L-1)/2 - 1 = q_R - 1$. □

Remark 7.3. The agreement is automatic but not vacuous: the two evaluations run through different instances of Proposition 4.3, with different $|B|$, different P_0, different V and, by Proposition 7.2, different q. It is therefore a genuine test of the two theorems against each other, and of their implementations. We checked it exactly on every splitting of every cycle length up to 60 (Section 11). For instance $\Theta(2, 2)$ has $q_\Theta = 0$, $n_2 = 2$, giving $(1, 2, 1)$, while $R(4)$ has $q_R = 1$, $n_0 = 1$, $x = 0$, also giving $(1, 2, 1)$ — by different bookkeeping.

Remark 7.4 (The classes are otherwise genuinely different). Away from the cycles the two answers differ, and not only by the shift in Proposition 7.1: the cyclomatic numbers differ, $c(R) = t$ against $c(\Theta) = t - 1$, and the second eigenvalue changes the nullity. For instance $\mathrm{In}\,A(L(R(4, 4, 4))) = (4, 3, 5)$ while $\mathrm{In}\,A(L(\Theta(4, 4, 4))) = (5, 2, 5)$, and $\mathrm{In}\,A(L(R(3, 3, 3))) = (3, 0, 6)$ while $\mathrm{In}\,A(L(\Theta(3, 3, 3))) = (4, 0, 5)$.

8 Multiplicity of the eigenvalue two

Reading the zero index of Theorems 5.1 and 6.2 gives closed formulas for $m_Q(G, 2)$.

Corollary 8.1. *For a rose, $m_Q(R, 2) = n_0 + n_2 - \mathbf{1}_{\{n_2 > 0\}} + \mathbf{1}_{\{n_2 = 0\}}\mathbf{1}_{\{x = 0\}}$. For a simple generalized theta,*

$$m_Q(\Theta, 2) = (n_0 - 1)_+ + (n_2 - 1)_+ + \mathbf{1}_{\{n_2 = 0\}}\mathbf{1}_{\{x_+ = 0\}} + \mathbf{1}_{\{n_0 = 0\}}\mathbf{1}_{\{x_- = 0\}}.$$

Moreover $m_Q(\Theta, 2) \leq c(\Theta)$ whenever $t \geq 3$, and $m_Q(\Theta, 2) = c(\Theta) + 1$ exactly for $t = 2$ with $\ell_1 + \ell_2 \equiv 0 \pmod 4$, that is exactly for the cycles of length $0 \pmod 4$.

Proof. The formulas are the zero indices. For the comparison recall $c(\Theta) = t - 1$. If $n_0, n_2 > 0$ then $m_Q = n_0 + n_2 - 2 \leq t - 2 < c + 1$. Suppose $n_0 > 0 = n_2$. If $n_1 \geq 1$ then $x_+ \geq n_0 > 0$ and $m_Q = n_0 - 1 \leq t - 1 = c$. If $n_1 = 0$ then $x_+ = n_0 - 2$, so the second term is nonzero only for $n_0 = 2$, where $m_Q = 2$ and $t = 2 + n_3$; this is $\leq c = 1 + n_3$ exactly when $n_3 \geq 1$, i.e. when $t \geq 3$, and equals $c + 1 = 2$ when $t = 2$. The case $n_2 > 0 = n_0$

is symmetric. If $n_0 = n_2 = 0$ then $m_Q \leq 2$, and $m_Q = 2$ forces $x_+ = x_- = 0$, hence $n_1 = n_3 = 1$ and $t = 2$. Collecting the $t = 2$ subcases, equality occurs exactly for the residue pairs $(0,0)$, $(2,2)$, $(1,3)$ and $(3,1)$, which are precisely the pairs with $\ell_1 + \ell_2 \equiv 0 \pmod 4$. □

Remark 8.2 (Relation to published bounds). Corollary 8.1 is a closed formula on two classes, not a bound on a general class, and should be read alongside two results of the latter kind. Zhao and Yu prove $m_Q(G,2) \leq c(G)+1$ for a connected graph *with a perfect matching*, with a characterisation of equality [6]. Li, Guo, Tian and Wang prove, for every connected graph,

$$m_Q(G,2) \leq c_2(G) + 1 \leq c(G) + 1,$$

where $c_2(G)$ is the minimum number of edges whose deletion destroys all even cycles [7]. Thus the bound $m_Q(\Theta,2) \leq c(\Theta)+1$, including the case without a perfect matching, is already a consequence of their general theorem. Our equality cases are the cycles of length $0 \pmod 4$, which are bipartite and do have a perfect matching, so they lie inside the hypothesis of [6] as well. What Corollary 8.1 adds on these classes is threefold:

- the exact value of $m_Q(G,2)$;
- its separation into a part counting kernel directions not adjacent to B, and a part recording the vanishing of an eigenvalue of P';
- the sharper class bound $m_Q(\Theta,2) \leq c(\Theta)$, and hence the failure of the extreme value $c(G)+1$, once $t \geq 3$.

The last comparison is with the general $c(G)+1$ benchmark; we do not claim that it improves the $c_2(G)+1$ bound case by case. We claim no priority over either paper.

9 Position relative to the cyclomatic bound

The conjecture

$$2s(L(G)) \leq c(G) + 1 \tag{7}$$

for finite connected simple graphs remains open. The weaker universal bound $s(L(G)) \leq c(G)$ is proved in [3], which records (7) as conjectural; neither is a claim of this paper. Both classes satisfy it, and they do so in the same quantitative way. Put $\Delta(G) = c(G) + 1 - 2s(L(G))$.

Some bound of this shape is needed. Akbari, Elphick, Kumar, Pragada and Tang conjectured that $s(L(G)) \leq 1$ for every connected G [8]; Francis and Uptain refute this and show that $s(L(G))$ is unbounded, so no constant bound survives [9]. A bound growing with $c(G)$, as in (7), is therefore the kind of statement that can still hold, and their examples are consistent with it: their family has $c = 3k$ and $s(L) = k+1$, so $\Delta = k - 1 \geq 0$.

Corollary 9.1 (Uniform slack). *Put*

$$S_R = \mathbf{1}_{\{n_2=0\}} \operatorname{sign}(x) \in \{-1, 0, 1\},$$

$$S_\Theta = \mathbf{1}_{\{n_2=0\}} \operatorname{sign}(x_+) + \mathbf{1}_{\{n_0=0\}} \operatorname{sign}(x_-) \in \{-2, \dots, 2\}.$$

Then

$$\Delta(R) = 3t - 1 - 2S_R, \qquad \Delta(\Theta) = 3t - 4 - 2S_\Theta,$$

and $S_\Theta = 2$ forces $t \geq 4$. In terms of the cyclomatic number,

$$\min_{c(R)=c} \Delta(R) = 3(c-1), \qquad \min_{c(\Theta)=c} \Delta(\Theta) = \begin{cases} 0, & c = 1, \\ 3, & c = 2, \\ 3c-5, & c \geq 3. \end{cases}$$

In both classes (7) holds, with equality only at $c = 1$, and there only for the cycles of length $1 \pmod 4$.

Proof. Substitute Corollaries 5.2 and 6.3 into $\Delta = c+1-2s$, using $c(R) = t$ and $c(\Theta) = t-1$. For the rose, $S_R \leq 1$ gives $\Delta \geq 3t-3 = 3(c-1)$. For the theta, $S_\Theta = 2$ needs $n_0 = n_2 = 0$, $n_1 \geq 2$ and $n_3 \geq 2$, hence $t \geq 4$; so $\Delta \geq 3t-8 = 3c-5$ for $t \geq 4$, while $S_\Theta \leq 1$ for $t \leq 3$ gives $\Delta \geq 3t-6$, namely 3 at $t = 3$ and 0 at $t = 2$. Equality $\Delta = 0$ therefore requires $c = 1$ in either class, that is a single cycle, and by Corollary 5.2 at $t = 1$ a cycle has $s(L(C_\ell)) = 1$ exactly when $\ell \equiv 1 \pmod 4$. ☐

Remark 9.2 (A restriction on extremal hosts). No rose with $t \geq 2$ and no generalized theta with $t \geq 3$ is extremal for (7), or even within 2 of extremal. Both classes admit a degree characterisation, which turns this into an exclusion of structural families rather than of two parametrised lists. A connected simple graph is a rose with $t \geq 2$ petals if and only if it has one vertex of degree at least 4 and every other vertex of degree 2. A connected simple graph is a generalized theta with $t \geq 3$ paths if and only if it is 2-connected and has exactly two vertices of degree at least 3, every other vertex having degree 2. So a minimal counterexample to (7) cannot itself be a rose with $t \geq 2$ or a generalized theta with $t \geq 3$, at any cyclomatic number. We claim no more than that. A graph obtained by attaching trees, or other structures, to such a core is not excluded: excluding it would need a result about attachments, which this paper does not prove.

The 2-connectivity hypothesis cannot be dropped. A graph consisting of two vertex-disjoint cycles joined by a path also has exactly two vertices of degree 3 and all others of degree 2, and it is not a generalized theta. Such dumbbell graphs lie outside both classes of this paper, and nothing proved here bears on them.

At $c = 2$ the classes treated here are the two-petal rose and the three-path generalized theta, and (7) holds on both with slack at least 3. A connected graph with $c(G) = 2$ may however carry pendant trees on its core, and nothing proved here applies to those, so the bicyclic case of (7) remains open.

That the exclusion is not vacuous can now be seen from outside. The 14-vertex graph of [9], two pentagons joined by bridges to adjacent vertices of a square, has $c = 3$ and $\mathrm{In}(A(L(G))) = (9, 0, 7)$, so $s(L(G)) = 2$ and $\Delta = 0$: an extremal host for (7) at $c = 3$. Corollary 9.1 places equality only at $c = 1$, and there is no conflict, because that corollary speaks only of roses and generalized thetas and this graph is neither. What the two statements say together is that extremal hosts exist above $c = 1$ and must be sought outside both classes, which is exactly the content of the exclusion above.

10 A partial extension to bridgeless cacti

A *bridgeless cactus* is a connected cactus in which every edge lies on a cycle; its blocks are cycles, and n_r now counts the blocks of length r mod 4. We record the following secondary result because it shows how far the one-boundary-vertex argument reaches, and exactly where it stops.

Proposition 10.1. *If G is a bridgeless cactus then*

$$s(L(G)) \le n_0 + n_1 - n_3 - \mathbf{1}_{\{n_0>0\}},$$

and consequently (7) *holds whenever* $n_0 + n_1 \le n_2 + 3n_3 + 1 + 2\,\mathbf{1}_{\{n_0>0\}}$.

Proof. Let a leaf cycle C of length ℓ meet the rest of the cactus H only at v; put $e = e_v$ and $M_H = Q(H) - 2I$. The elimination of Lemma 4.1, with the data of Lemma 3.1 at $s = +1$, gives the exact signature recurrences

$$\begin{aligned}
\ell \equiv 3\ (4): &\quad \operatorname{sig}(Q(G) - 2I) = \operatorname{sig} M_H,\\
\ell \equiv 1\ (4): &\quad \operatorname{sig}(Q(G) - 2I) = \operatorname{sig}(M_H + 4ee^T),\\
\ell \equiv 0\ (4): &\quad \operatorname{sig}(Q(G) - 2I) = \operatorname{sig}(M_H + 2ee^T),\\
\ell \equiv 2\ (4): &\quad \operatorname{sig}(Q(G) - 2I) = \operatorname{sig}(M_H - v),
\end{aligned}$$

the 0 mod 4 branch retaining in addition one zero direction.

Only the fourth branch needs comment. Eliminate the range part of the path, and order the remaining coordinates as v, then the coupled path-kernel direction, then $V(H) \setminus \{v\}$. The residual matrix is

$$K = \begin{pmatrix} a & \mu & r^T \\ \mu & 0 & 0 \\ r & 0 & M_H - v \end{pmatrix}, \qquad \mu \neq 0.$$

Its leading block $J = \left(\begin{smallmatrix} a & \mu \\ \mu & 0 \end{smallmatrix}\right)$ is invertible, with $\operatorname{In} J = (1, 0, 1)$ and $(J^{-1})_{11} = 0$. Eliminating the r-coupling therefore leaves $M_H - v$ unchanged, so K is congruent to $J \oplus (M_H - v)$.

Root the block–cut tree at a 0 mod 4 cycle if one exists and at any cycle otherwise, and build outward. A positive rank-one update raises the signature by at most two and deleting a principal vertex raises it by at most one, so the four residue classes cost at most $2, 2, 1, 0$ for $0, 1, 2, 3$ respectively. For the initial cycle, Corollary 5.2 at $t = 1$ gives $s(L(C)) = 0, 1, 0, -1$ in the same residue order, so rooting at a 0 mod 4 cycle saves the full two-unit cost for one block and otherwise one unit. Hence $\operatorname{sig}(Q(G) - 2I) \le 2n_0 + 2n_1 + n_2 - 1 - \mathbf{1}_{\{n_0>0\}}$, and $c(G) = n_0 + n_1 + n_2 + n_3$; Lemma 2.2 gives the first inequality and comparing twice its right-hand side with $c(G) + 1$ gives the stated sufficient condition. □

Remark 10.2 (The exact stopping point). Proposition 10.1 is not a complete inertia formula for cacti, and it is worth saying precisely what stops the method rather than the proposition. Proposition 4.3 needs only that the components of $G - B$ be paths attached to B at their two ends, so it does apply to some cactus decompositions. What does not carry over is the closed evaluation.

Two things obstruct it. First, a cactus block may meet the rest of the graph at more than two articulation vertices, so the boundary space can have dimension greater than 2 and the analysis of Sections 5 and 6 does not cover it. Second, even with exactly two articulation vertices there is in general no automorphism exchanging them, so P' has no distinguished eigenbasis and the two-dimensional evaluation of Section 6 is unavailable. Concretely, in an induction aimed at (7) the 0 mod 4 and 1 mod 4 leaf branches require root response inequalities: with $M_H y = e_v$ and $g_v = e_v^T y$, the unresolved thresholds are

$$1 + 2g_v \ge 0 \qquad \text{and} \qquad 1 + 4g_v \ge 0$$

respectively, in the range-compatible case. Establishing these for all relevant equality hosts would be a new global result and is not attempted here.

The bridgeless hypothesis is likewise not removable. The extremal graph of [9] recalled in Remark 9.2 is a cactus, and it attains $\Delta = 0$ at $c = 3$; its three cycles are joined by bridges, so it lies outside Proposition 10.1. Whatever governs equality for cacti therefore has to be sensitive to bridges, which the bridgeless argument given here cannot see.

11 Exact computational verification

The proofs above are symbolic and do not rely on enumeration. Independently of them, every formula-bearing result of this paper, together with every singular branch of Lemma 3.1, was checked by exhaustive computation over bounded families, in exact rational or integer arithmetic with no floating point, and by two procedures that share no implementation. The first is a symmetric congruence over the rationals, valid on singular and indefinite blocks, which is what the even-length blocks require. The second computes the characteristic polynomial over the integers and counts its positive and negative roots by the sign variations of the coefficient sequence. For a polynomial whose roots are all real that count is an equality rather than a bound, so it is exact for the characteristic polynomial of a symmetric matrix. This is the classical rule of signs together with one observation, and we record the argument because the second procedure rests on it. Write $p(x) = x^k q(x)$ with $q(0) \neq 0$ and $\deg q = m$, and let $V(q)$ be the number of sign variations in the coefficient sequence of q. Descartes' rule gives $V(q) \geq P$ and $V(q(-x)) \geq N$, where P and N count the positive and the negative roots of q with multiplicity. Since $q(0) \neq 0$, the two variation counts satisfy $V(q) + V(q(-x)) \leq m$. Indeed, between consecutive nonzero coefficients whose exponents differ by d, the two sequences contribute one variation in total when d is odd and either zero or two when d is even; in every case the contribution is at most d, and summing the exponent gaps gives m. If every root of p is real then $P + N = m$, so

$$m = P + N \leq V(q) + V(q(-x)) \leq m,$$

and both inequalities are equalities: $V(q) = P$ and $V(q(-x)) = N$. The nullity is k. No discrepancy was found in any family.

The families checked were:

- all path blocks of length at most 60, for both signs;
- all roses with at most five petals of length at most 12;
- 11 975 distinct simple generalized theta graphs in five declared, overlapping windows, reaching seven paths, path length 36, and order 110. The five sweeps perform 12 826 checks. The supplement gives the exact window definitions and tabulates generated, order-excluded, duplicate, and new multisets for each window;
- bridgeless cacti with at most three blocks.

One theta sweep used only lengths between 21 and 36, so that the dependence on residues modulo 4 is tested away from the smallest representative of each residue class.

The minimum slack of Corollary 9.1 was attained for every number of paths and petals tested. The only graphs with $\Delta = 0$ found in any family were C_5 and C_9, as Corollary 9.1 requires.

One check invokes a published theorem rather than the machinery of this paper. Applying the Wang–Fan inequality $-c_3(G) \leq s(G) \leq c_5(G)$ [1] to $G = L(\Theta)$, with the cycles of the line graph enumerated exactly, every case lies inside that bound and the values of Corollary 6.3 attain it in both directions.

Per-object counts, the scripts, and the archived verification outputs are in the reproducibility supplement. These finite checks test the implementation and the singular branches; they neither replace the proofs nor bear on novelty.

12 Related work

The methods are not presented as new. The incidence identity (1), the generalized Schur complement of Lemma 4.1, the saddle-point inertia of Lemma 4.2 and the path-response technique are all standard. The central result is Theorem 6.2, whose prior-art status is qualified below, together with the observation that it and Theorem 5.1 are the cases $p = 1, 2$ of one reduction, and the exact consequences that observation makes available: Proposition 7.1, the uniform slack of Corollary 9.1, and the restriction on extremal hosts in Remark 9.2.

What the prior-art review found

A targeted review located no source stating an exact mixed-residue formula for the full inertia of $Q(\Theta) - 2I$ for an arbitrary number of paths. Given the scope of that review, recorded under *Limitations* below, we do not claim that no proper part of Theorem 6.2 appears in the literature; parts of it plausibly do, and Section 12 names the closest candidates we found. Adjacency inertia of line graphs is determined for trees [4] and bounded in general by cycle-residue counts [1]; line-graph nullity is determined for unicyclic graphs of depth one [5], which intersects only the cycles, where the two classes of this paper meet. For $m_Q(G, 2)$ see Remark 8.2.

For the rose class the published antecedents are more substantial and cover proper parts of Theorem 5.1. Wang and Fan obtain, among other results, the two-petal case in which both lengths are $2 \pmod 4$ [1]. Li, Zhang and Wang give exact signless-Laplacian characteristic polynomials for equal-petal Dutch windmill graphs [10], from which the inertia of those roses can be read off. Their D_p^q is our $R(p, \ldots, p)$ with q petals. Their Theorem 9 splits the characteristic polynomial into $q - 1$ copies of a path factor and one quotient factor, matching the local-path plus scalar decomposition used here. At the threshold 2, their factorization gives nullity q for $p \equiv 0 \pmod 4$, nullity $q - 1$ for $p \equiv 2 \pmod 4$, and nullity zero for odd p, exactly the four equal-petal specializations of Theorem 5.1; the signs of the remaining scalar factor give the same positive and negative indices. Thus the equal-petal overlap is an antecedent, not a novelty claim of this paper. He and van Dam study Laplacian spectral characterization of arbitrary roses [11], and Abdian, Ashrafi and Brunetti prove that p-roses with $p \geq 3$ are determined by their signless-Laplacian spectrum [12]. Those last two are inverse-spectral in their stated aims: they ask whether a spectrum determines the graph, rather than counting the eigenvalues of $Q(R)$ above, at and below 2. The seven-page paper of Abdian, Ashrafi and Brunetti was examined in full. It uses signless-Laplacian spectral invariants to solve an inverse problem, but does not compute the inertia of $Q(R) - 2I$ or $A(L(R))$, nor state a mixed modulo-four threshold formula. What Theorem 5.1 provides, in the formulation developed here, is the arbitrary mixed-residue case with every singular branch and every zero multiplicity. The comparison with [11] remains at the level stated in its verified record.

Citation chasing from the spectral-characterization literature located two earlier adjacency-spectral studies of three-path theta graphs [13, 14]. They ask whether the adjacency spectrum determines the graph; neither studies the inertia of the line graph or the distribution of the signless-Laplacian spectrum about 2. Three further sources treat the same graph families, or the same matrix, but a different question. Liu and Lu give a Laplacian spectral characterization of dumbbell and theta graphs [15]. At three paths their class is exactly ours. Their matrix is the Laplacian, however, and their question is determination by the spectrum rather than the distribution of the spectrum about a threshold. Liu and Wang study the signless Laplacian spectral radius [16]: the matrix is ours, but the invariant is the largest eigenvalue and theta graphs enter as the forbidden subgraph of an extremal problem rather than as the object whose spectrum is computed. Guo and Wang give a (signless) Laplacian spectral characterization of line graphs of lollipop graphs [17], in which line graphs and the signless Laplacian both appear, again with spectral determination as the question. None of the three counts eigenvalues above, at and below 2.

One line of work counts eigenvalues of Q against the same threshold 2 that we use, and is therefore closer to the present question than anything above. Wang, Belardo, Wang and Huang treat the graphs having exactly three signless-Laplacian eigenvalues at least 2 [18]. Wang and Belardo determine the graphs with exactly one or two Q-eigenvalues greater than or equal to 2 [19], as stated in their abstract; Xu and Zhou [20] report the same work as determining all connected graphs with at most two signless-Laplacian eigenvalues exceeding 2. Feng, Wang and Belardo study a class at the same threshold from the standpoint of spectral characterization [21].

The two descriptions of [19] count different quantities in the notation of this paper: eigenvalues greater than or equal to 2 is $n_+(Q - 2I) + n_0(Q - 2I)$, whereas eigenvalues exceeding 2 is $n_+(Q - 2I)$ alone.

These are inverse problems: the number of eigenvalues above the threshold is prescribed, and the graphs realising it are determined. Theorems 5.1 and 6.2 run the other way, computing that number, and the other two indices, for a prescribed class and as a function of the residues. We have examined [20] in full. For [18] and [19], the official abstracts and zbMATH records were checked, but the full texts were not available to the authors. A theorem-by-theorem comparison with these two papers is therefore incomplete. They remain the natural place to look for overlap at small values of $n_+(Q - 2I)$, and no novelty is claimed for any subcase that may overlap their classifications.

For roses in particular, Ma and Huang give a signless-Laplacian spectral characterisation of 4-roses [22], and Liu and Huang a Laplacian one for 3-roses [23]. Both fix the number of petals and ask whether the spectrum determines the graph, rather than computing an inertia for arbitrary t.

Two distinctions are worth making explicit. Liu defines both infinity and three-path theta graphs, but the bicyclic results of [24] are restricted to the class $\mathcal{B}_n^{++}$: infinity graphs whose two core cycles share one vertex, with attached trees. They do not cover the adjacency inertia of three-path theta graphs. In any event, the object of this paper is $A(L(\Theta))$, rather than the adjacency matrix of Θ itself. Separately, [25] studies the inertia of the *distance* matrix of line graphs of unicyclic graphs, not of the adjacency matrix, and so does not bear on the present question.

13 Limitations

No claim of absolute novelty is made. The review rests on abstracts and verified metadata for the two inaccessible works identified above, was conducted in English, and used zbMATH Open but not MathSciNet because no institutional MathSciNet access was available. Searches included "theta graph", "generalized theta", "multi-theta", "multiple-path graph", "signless Laplacian", "Q-spectrum", "eigenvalue two", "threshold two", "line graph inertia", and combinations of these terms. Negative searching cannot prove absolute novelty, and any overlap found later must be recorded and the contribution restated.

The conjecture (7) remains open. Corollary 9.1 verifies it on both classes with quantified slack, and Remark 9.2 restricts where extremal hosts can live, but neither is a proof and neither is presented as one.

Finally, Proposition 4.3 is stated for any B whose removal leaves paths attached at their two ends, but it is *evaluated* only at $p = 1$ and $p = 2$. Both evaluations use that each block kernel is at most one-dimensional, and the one at $p = 2$ uses an automorphism to fix the eigenvectors of P'. Remark 10.2 explains why the cactus case is not a corollary: the boundary may have dimension greater than 2, and even at dimension 2 the two vertices need not be exchangeable. A composable description for general boundaries is not attempted here.

Data and reproducibility

No empirical data are used. The scripts that perform the verification of Section 11, together with their archived verification outputs and the per-object counts, form the reproducibility supplement accompanying this paper. It is a single archive, distributed with Version 1.0 of this paper and named

`Paone-Paone_Inertia-of-Line-Graphs-of-Rose-and-Generalized-Theta-Graphs_source-and-reproducibility-v1.0.zip`.

It contains the LaTeX source of this article, its BibTeX database, generated bibliography, and a path-sanitised build log; citation metadata and the CC BY 4.0 licence; the verification scripts under `reproducibility/code/`, of which `verify_unified.py` is the entry point and `reconcile_theta_domain.py` reproduces the window table of Section 11; the frozen outputs under `reproducibility/results/`; an independently written exact checker and its report in a dedicated directory; and integrity manifests and SHA-256 checksums for the released payload. The scripts require only Python 3.10 or later and no external library.

Version 1.0 and its accompanying archive are distributed through the Zenodo record identified by DOI 10.5281/zenodo.21744051. A reader who holds this PDF without the archive should obtain it from that record.

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During the research and preparation of this work, the authors used OpenAI ChatGPT and Anthropic Claude services, including Claude Code, for exploratory analysis, code development and checking, literature triage, manuscript review, and language editing. These tools produced suggestions and executable code; they were not treated as proofs or sources of scientific authority.

The authors checked the mathematical statements, proofs, citations, and computational results against the manuscript's arguments, primary bibliographic records, and the accompanying reproducibility materials. The extent of full-text literature review is stated in Section 12. The authors revised and approved the final manuscript and take full responsibility for its content. Generative-AI systems were not credited as authors and did not make autonomous research or publication decisions.

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